Prime–Semiprime Balance and a Denominator-Four Threshold in Prime-Producing Sieves

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September 2026

Abstract

We record an exact one-parameter specialization of the Ford–Maynard criterion for when Type I/Type II information guarantees a prime asymptotic. For $1/2 < \beta < 1$, set

$$\tau_\beta = \frac{\beta}{1+\beta}, \quad \alpha_\beta = \frac{\beta}{e+\beta}, \quad \gamma_\beta = \frac{1}{1+\beta},$$

and consider $P_\beta = (\gamma_\beta, \alpha_\beta, \tau_\beta - \alpha_\beta)$. We prove that Ford and Maynard's rational-cover conditions hold for this family if and only if $\beta \leq e/3$. The binding obstruction is the denominator-four grid point $1/4$; denominators $n \geq 5$ are automatically covered on the admissible side. The parameter α_β is motivated independently by a model balance between prime and β-unbalanced semiprime mass. For $\beta = 0.9$, the shape exponent in Li and Liu's 2026 theorem $(1 + 1.9)$, the associated parameters are $\tau = 9/19$ and $\gamma = 10/19$, with $10/19 - 1/2 = 1/38$. This is a parameter-level correspondence only: no new case of the binary Goldbach conjecture is claimed.

Keywords. sieve methods; Type I/Type II sums; prime-producing sieves; Goldbach conjecture; almost primes.
2020 Mathematics Subject Classification. 11N35, 11P32.

1 Introduction

The binary Goldbach conjecture asks whether every even integer greater than 2 is a sum of two primes, with the expected count of representations governed by the Hardy–Littlewood singular series [3]. Chen's theorem replaces one prime by an integer with at most two prime factors. Li and Liu recently introduced an interpolation: Proposition $(1 + a)$ asserts that every sufficiently large even N has

$$N = p + rq, \qquad r \leq q^{a-1},$$

where p, q are prime and r is either 1 or prime; they prove Proposition $(1 + 1.9)$ unconditionally [2].

Ford and Maynard developed a general theory determining when specified Type I and Type II information forces an asymptotic for primes in a nonnegative sequence [1]. Their criterion is expressed through rational-cover conditions on (γ, θ, ν).

This note records a one-parameter specialization linking an unbalanced-semiprime shape parameter β to that rational-cover geometry. A natural model balance parameter

$$\alpha_\beta = \frac{\beta}{e+\beta}$$

meets the denominator-four obstruction at exactly $\beta = e/3$. The model motivation and the rigorous Ford–Maynard corollary are kept logically separate. No unconditional Goldbach theorem follows.

2 Ford–Maynard's criterion

We use Ford and Maynard's notation [1, Theorem 2.2]. In the branch relevant here their reduced parameter set $\mathcal{Q}$ includes triples satisfying

$$\frac{1}{2} \le \gamma \le 1-\theta-\nu, \qquad 0 \le \theta < \theta+\nu \le \frac{1}{2}.$$

Let $M = \lfloor 1/(1-\gamma) \rfloor$. Their conditions are

$$\begin{aligned} &(\mathrm{A1}): \quad \forall n \ge M+1,\ \exists a \in \mathbb{N}:\ a/n \in [\theta, \theta+\nu], \\ &(\mathrm{A2}): \quad \exists h \in \mathbb{N}:\ h(1-\gamma) \in [\theta, \theta+\nu] \cup [1-\theta-\nu, 1-\theta]. \end{aligned}$$

If both hold, the prime asymptotic follows under their Type I, Type II and growth hypotheses. If either fails, their theorem constructs sequences showing that those data do not force the asymptotic [1, Theorem 2.2].

3 Semiprime geometry and the balance parameter

Let $1/2 < \beta < 1$ and suppose

$$n = qr \asymp N, \qquad q \le r, \qquad q \le r^\beta.$$

Writing $q = N^{u+o(1)}$, $r = N^{v+o(1)}$ gives $u + v = 1 + o(1)$ and $u \le \beta v$, hence

$$u \le \frac{\beta}{1+\beta} + o(1).$$

Set

$$\tau_\beta := \frac{\beta}{1+\beta}.$$

For $\beta = 0.9$, this is $9/19$, matching the factor-shape boundary of Proposition $(1 + 1.9)$.

A model relative mass for semiprimes with smaller-factor exponent in $[\alpha, \tau_\beta]$ is

$$I(\alpha, \beta) = \int_\alpha^{\beta/(1+\beta)} \frac{dt}{t(1-t)} = \log\left(\beta \frac{1-\alpha}{\alpha}\right).$$

The balance equation $I = 1$ gives

$$\alpha_\beta := \frac{\beta}{e+\beta}.$$

This is a model-density motivation, not an unconditional Goldbach asymptotic. Finally set

$$\gamma_\beta := \frac{1}{1+\beta} = 1 - \tau_\beta, \qquad \nu_\beta := \tau_\beta - \alpha_\beta.$$

4 The $e/3$ threshold

Theorem 4.1 (Parameterized rational-cover threshold)**.** *For* $1/2 < \beta < 1$*, let*

$$P_\beta = \left(\frac{1}{1+\beta}, \frac{\beta}{e+\beta}, \frac{\beta}{1+\beta} - \frac{\beta}{e+\beta} \right).$$

Then $P_\beta \in \mathcal{Q}$*. Ford–Maynard conditions* (A1) *and* (A2) *hold for* P_β *if and only if*

$$\boxed{\beta \le \frac{e}{3}}.$$

Consequently, if the full Ford–Maynard Type I, Type II and growth hypotheses hold for P_β*, their asymptotic conclusion follows for* $1/2 < \beta \le e/3$*. For* $e/3 < \beta < 1$*, the corresponding Type I/II information does not, in their general framework, force the prime asymptotic.*

Proof. For $1/2 < \beta < 1$,

$$\frac{1}{3} < \tau_\beta < \frac{1}{2}, \qquad \frac{1}{2} < \gamma_\beta < \frac{2}{3}, \qquad 0 < \alpha_\beta < \tau_\beta.$$

Also $1 - \alpha_\beta - \nu_\beta = 1 - \tau_\beta = \gamma_\beta$, so P_β lies in the relevant branch of $\mathcal{Q}$.

Since $1 - \gamma_\beta = \tau_\beta$,

$$\frac{1}{1-\gamma_\beta} = \frac{1+\beta}{\beta} = 1 + \frac{1}{\beta} \in (2,3),$$

so $M = 2$, and (A1) concerns every integer $n \ge 3$. Condition (A2) holds with $h = 1$, because $1 - \gamma_\beta = \tau_\beta \in [\alpha_\beta, \tau_\beta]$.

For $n = 3$: $\tau_\beta \ge 1/3 \iff \beta \ge 1/2$, true since $\beta > 1/2$ strictly, and $\alpha_\beta \le 1/3 \iff \beta \le e/2 \approx 1.359$, true throughout $\beta < 1$. Hence $1/3 \in [\alpha_\beta, \tau_\beta]$ for every β in range, so (A1) holds unconditionally at $n = 3$.

For $n = 4$, since $\tau_\beta < 1/2$, the only positive grid point $a/4$ that can lie in the interval is $1/4$. Hence

$$\alpha_\beta \le \frac{1}{4} \iff \frac{\beta}{e+\beta} \le \frac{1}{4} \iff 3\beta \le e \iff \beta \le \frac{e}{3}.$$

This proves necessity.

Assume $1/2 < \beta \le e/3$. For $n = 5$, if $\beta \le e/4$, then $\alpha_\beta \le 1/5$ and $1/5 < \tau_\beta$, so $1/5$ works. If $e/4 < \beta \le e/3$, then $\beta > e/4 > 2/3$, hence $\tau_\beta > 2/5$, while $\alpha_\beta \le 1/4 < 2/5$; thus $2/5$ works.

For $n \ge 6$, write

$$L(\beta) := \tau_\beta - \alpha_\beta = \frac{\beta(e-1)}{(1+\beta)(e+\beta)}.$$

Its derivative has the sign of $e - \beta^2$, so L is increasing on $(1/2, 1)$. Therefore

$$L(\beta) > \lim_{t \downarrow 1/2} L(t) = \frac{e-1}{3e + 3/2} \approx 0.17797 > \frac{1}{6}.$$

For every $n \ge 6$, $[n\alpha_\beta, n\tau_\beta]$ has length $nL(\beta) > 1$ and therefore contains an integer a. Thus $a/n \in [\alpha_\beta, \tau_\beta]$, proving (A1).

If $e/3 < \beta < 1$, then $\alpha_\beta > 1/4$ and $\tau_\beta < 1/2$, so (A1) fails already at $n = 4$. The Ford–Maynard conclusions now follow from their Theorem 2.2. $\square$

Remark 4.2 (Endpoint $\beta = 1$)**.** *Theorem 4.1 excludes $\beta = 1$. At that endpoint $\tau_1 = 1/2$, so $2/4 = 1/2$ re-enters the Type II interval. With $\alpha_1 = 1/(e+1)$ and $\gamma_1 = 1/2$, conditions (A1) and (A2) are again satisfied. Thus this one-parameter path exhibits re-entry at the endpoint, consistent with the discontinuity phenomena in Ford–Maynard's theory.*

5 The Li–Liu $1 + 1.9$ specialization

Li and Liu define Proposition $(1 + a)$ by $N = p + rq$ with $r \leq q^{a-1}$ and prove Proposition $(1 + 1.9)$ unconditionally [2, Theorem 1.1]. Thus $\beta = a - 1 = 0.9$, and

$$\tau_{0.9} = \frac{9}{19}, \qquad \gamma_{0.9} = \frac{10}{19}, \qquad \alpha_{0.9} = \frac{0.9}{e + 0.9} \approx 0.2487368433.$$

Moreover

$$0.9 < \frac{e}{3} \approx 0.9060939428, \qquad \frac{e}{3} - 0.9 \approx 0.0060939428,$$

and

$$\frac{10}{19} - \frac{1}{2} = \frac{1}{38}.$$

The last identity is a parameter comparison, not a statement that only 1/38 of a proof of Goldbach remains. Ford–Maynard require full Type I/Type II estimates for broad coefficient classes; distribution results with larger numerical exponents may impose factorability or coefficient restrictions and cannot be substituted solely by comparing exponents.

6 Interpretation of the repeated $e/3$ threshold

The constant $e/3$ appears in two descriptions of the same parameterized family. The model balance equation gives $\alpha_\beta = \beta/(e + \beta)$, and $\alpha_\beta = 1/4$ is equivalent to $\beta = e/3$. In the Ford–Maynard criterion, $M = 2$, so (A1) concerns every $n \geq 3$; the denominator $n = 3$ is always satisfied, and $n = 4$ is the first (and, throughout the whole range $1/2 < \beta < 1$, the only) denominator that can fail. Because $\tau_\beta < 1/2$, the relevant grid point there is exactly 1/4. Hence the same equality is the exact rational-cover transition.

These are not logically independent theorems, because both use the chosen parameter α_β. The synthesis is nevertheless informative: a parameter motivated by prime-versus-unbalanced-semiprime model mass meets the first exact rational obstruction of a general prime-producing sieve at the same value.

7 Limitations and Goldbach relevance

The result is intentionally limited.

1. The binary Goldbach conjecture remains open.
2. We do not establish the Ford–Maynard Type I/Type II hypotheses for the Goldbach complement sequence at P_β.
3. Li–Liu's theorem $(1 + 1.9)$ is an almost-prime representation theorem; it does not by itself supply the arbitrary-coefficient Type I/II information required here.

4. The integral defining α_β is a model mass-balance calculation, not an unconditional binary Goldbach asymptotic.

5. A numerical distribution exponent exceeding 10/19 is insufficient unless its coefficient class, factorability and residue uniformity match the Ford–Maynard hypotheses.

6. We make a conservative priority statement only: in the literature searches undertaken for this note, we did not locate this explicit $e/3$ one-parameter specialization or the $0.9 \mapsto (9/19, 10/19, 1/38)$ interpretation.

8 Conclusion

For

$$P_\beta = \left(\frac{1}{1+\beta}, \frac{\beta}{e+\beta}, \frac{\beta}{1+\beta} - \frac{\beta}{e+\beta} \right), \qquad \frac{1}{2} < \beta < 1,$$

Ford–Maynard's rational-cover criterion has the exact transition $\beta = e/3$. The obstruction is denominator four: the point 1/4 exits the Type II interval precisely when $\beta > e/3$, while all larger required denominators remain covered on the admissible side. The Li–Liu $1 + 1.9$ shape corresponds to $\beta = 0.9 < e/3$ and to the complementary exponents 9/19 and 10/19.

The result does not resolve Goldbach. Its role is to isolate an exact parameter transition connecting unbalanced almost-prime geometry with the optimality theory of Type I/Type II prime-producing sieves.

Research disclosure

This work was developed within EM Foundation Research. Artificial-intelligence systems were used for exploratory symbolic calculations, literature navigation, and adversarial review. The author selected the claims, reconstructed the proof from the cited primary sources, and assumes responsibility for the manuscript.

References

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